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Correlated Majority Model

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We extend the bichromatic majority model by including (one-dimensional isotropic) correlations and numerically discuss, through Monte Carlo simulations, the simple, 1/3, and 2/3 majority rules. We calculate, as functions of the concentration ρ and correlation degree κ, the mean finite cluster size ξ and the order parameter m as well as their respective critical exponents ν and β. We find $\nu \simeq \beta \simeq 1$ regardless of the correlation degree or the type of majority. Also, a supplementary divergence of ξ is observed at the $\kappa > 0$ borderline.

KEY WORDS: Majority model; geometrical correlations; Monte Carlo; critical phenomena.

1. INTRODUCTION

Many problems in physics and other branches of knowledge relie on majority arguments. Various real-space renormalization groups and even a recent modelization of "political" behavior[(1)] belong to this category. Along this line, a specific geometrical model *(majority model)* was introduced in 1982.[(2)] The model admits the presence of colored plaquettes which can choose among q colors. The $q = 2$ model *(bichromatic model)* was treated within a renormalization group framework in ref. 2, and within a Monte Carlo framework in ref. 3. The *polychromatic* model (arbitrary integer q) has also been studied[(4)] (within the real-space renormalization group). However, all these studies share the basic hypothesis of *no correlation* of the colors chosen by neighboring plaquettes (distributed on a d-dimensional array such as, for instance, the simple hypercubic lattice). In the present work we study, within Monte Carlo simulations, the bichromatic model by allowing correlations between first-neighboring plaquettes on a

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linear chain ($d = 1$). More specifically, the correlation is assumed *isotropic*, i.e., the probability of having a given color at the right of any plaquette equals the probability of having it at its left. Although the model is formulated in terms of the p_0-majority ($0 \leqslant p_0 \leqslant 1$), we have run simulations only for $p_0 = 1/2$ (simple majority), 1/3, and 2/3.

The model and the results are presented in Section 2 and our conclusions in Section 3.

2. MODEL AND RESULTS

We consider, for a given choice of p_0, a one-dimensional L-sized strip of "black" and "white" stochastically chosen plaquettes. Any pair of first-neighboring plaquettes is assumed to satisfy the following distribution law:

$$\begin{aligned} P(\sigma_i, \sigma_{i+1}) = {} & p_{\mathrm{WW}} \delta(\sigma_i + 1)\, \delta(\sigma_{i+1} + 1) + p_{\mathrm{WB}} \delta(\sigma_i + 1)\, \delta(\sigma_{i+1} - 1) \\ & + p_{\mathrm{BW}} \delta(\sigma_i - 1)\, \delta(\sigma_{i+1} + 1) + p_{\mathrm{BB}} \delta(\sigma_i - 1)\, \delta(\sigma_{i+1} - 1) \end{aligned} \tag{1}$$

where i runs over the L plaquettes, $\sigma_i = -1$ ($\sigma_i = +1$) corresponds to "white" ("black") color, and

$$p_{\mathrm{WW}} + p_{\mathrm{WB}} + p_{\mathrm{BW}} + p_{\mathrm{BB}} = 1 \tag{2}$$

We focus the *isotropic* case, i.e., $p_{\mathrm{WB}} = p_{\mathrm{BW}}$; hence

$$p_{\mathrm{WW}} + 2p_{\mathrm{BW}} + p_{\mathrm{BB}} = 1 \tag{3}$$

We define the *correlation degree*

$$\kappa \equiv \langle \sigma_i \sigma_{i+1} \rangle_P - \langle \sigma_i \rangle \langle \sigma_{i+1} \rangle_P = p_{\mathrm{BB}} + p_{\mathrm{WW}} - 2p_{\mathrm{BW}} - (p_{\mathrm{BB}} - p_{\mathrm{WW}})^2 \tag{4}$$

and the *black concentration*

$$\rho \equiv p_{\mathrm{BB}} + p_{\mathrm{BW}} \tag{5}$$

Equations (3)–(5) immediately yield

$$\begin{aligned} p_{\mathrm{WW}} &= (1-\rho)^2 + \kappa/4 \\ p_{\mathrm{WB}} &= p_{\mathrm{BW}} = \rho(1-\rho) - \kappa/4 \\ p_{\mathrm{BB}} &= \rho^2 + \kappa/4 \end{aligned} \tag{6}$$

The zero-correlation case ($\kappa = 0$) yields $p_{\mathrm{WW}} = (1-\rho)^2$, $p_{\mathrm{WB}} = p_{\mathrm{BW}} = \rho(1-\rho)$, and $p_{\mathrm{BB}} = \rho^2$; hence ρ recovers the variable p of ref. 3. Also, since

p_{WW}, p_{BW}, and p_{BB} must belong to the interval [0, 1] and must satisfy Eq. (3), Eqs. (6) fully determine the physically allowed region in the (ρ, κ) space: see Fig. 1.

To perform the computational simulations [for a given pair (ρ, κ)] we proceed as follows. A strip configuration is fully determined by the sequence of colors of its L plaquettes. We generate a random number $r_1 \in [0, 1]$: if $r_1 \leqslant \rho$, then the first plaquette is black, if $r_1 > \rho$ then the first plaquette is white. We generate a second random number $r_2 \in [0, 1]$: if the first plaquette is black and $r_2 \leqslant p_{BB}/(p_{BB} + p_{BW})$ $[r_2 > p_{BB}/(p_{BB} + p_{BW})]$, then the second plaquette is black (white); if the first plaquette is white and $r_2 \leqslant p_{WB}/(p_{WW} + p_{WB})$ $[r_2 > p_{WB}/(p_{WW} + p_{WB})]$, then the second plaquette is black (white). This same procedure is followed up to the last of the L plaquettes by considering, in order to determine the color of the ith plaquette, the actual color of the $(i-1)$th plaquette. We repeat the entire algorithm N_0 times in order to have N_0 different strip configurations. We have typically used $L = 30{,}000$ and $N_0 = 10{,}000$.

Let us now describe the quantities we measure for each one of the N_0 strip configurations. We call l the strip size at which we lose (if we do) the black p_0-majority *assumed to exist*. To check the majority, we start considering the plaquette which is at the center of the strip [i.e., $i =$ integer $(L/2)$]. If its color is white, we abandon the experiment and this configuration is not included among the N_0 ones with which we shall calculate averages. If its color is black, we consider the cluster of three plaquettes

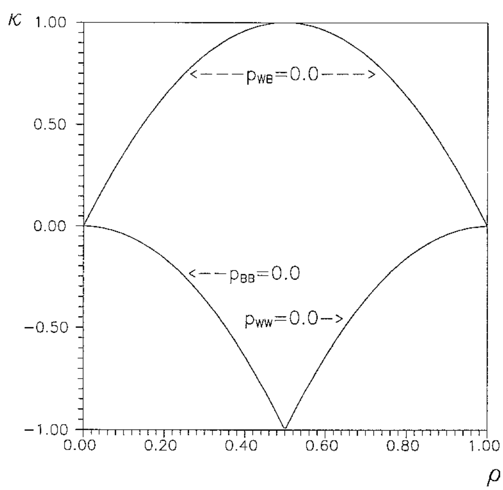

Fig. 1. The physically allowed region in the ρ (black concentration) versus κ (correlation degree) space; the external region is forbidden.

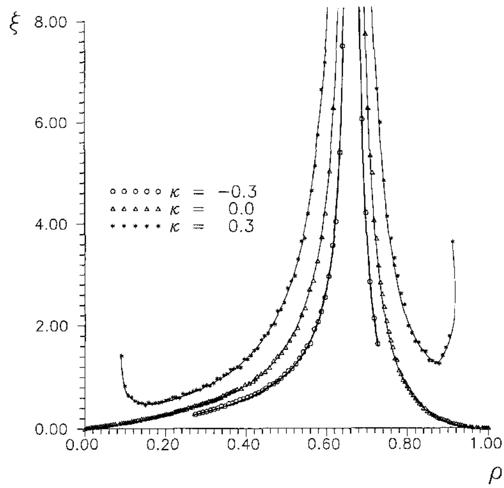

Fig. 2. The ρ dependence (for increasing ρ) of the black dominating mean finite-cluster size for typical values of the correlation degree κ for $p_0 = 2/3$ (the figures for $p_0 = 1/3$ and $p_0 = 1/2$ are very similar). The curves are guides to the eye. Here $L = 30{,}000$ and $N_0 = 10{,}000$ were used.

constituted by the central plaquette plus its two first-neighboring ones. If we now lose the black p_0-majority, then $l = 3$; if not, we continue and construct the five-plaquette cluster by adding to the previous cluster the new two first-neighboring plaquettes. If the black majority is maintained up to the L-sized cluster, we refer to this case as to "infinite" cluster; if the black

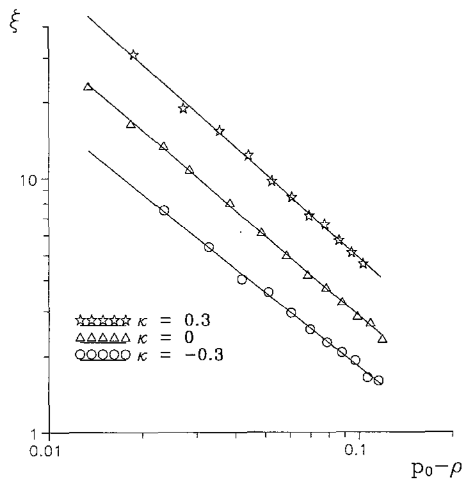

Fig. 3. log ξ versus log $|\rho - p_0|$ for $p_0 = 2/3$ and typical values of κ (here $L = 30{,}000$ and $N_0 = 10{,}000$) and $\rho \to p_0 - 0$. The case $\rho \to p_0 + 0$ is very similar.

majority is lost at some previous level, we refer to it as to "finite cluster." The *black-dominating mean finite-cluster size* is defined through $\xi \equiv \langle l \rangle_{\text{finite cluster}}$, the average being done on the set of the N_0 experiments. The *order parameter m* is defined as the proportion of N_0 experiments for which we succeeded in maintaining the black p_0-majority up to arrival at the L-sized strip.

The reversal of the majority appears as a second-order critical phenomenon. For fixed κ, the transition occurs at $\rho = p_0$ if ρ is *increasing* (from its minimal value to its maximal value: see Fig. 1, and it occurs at $\rho = 1 - p_0$ if ρ is *decreasing* (from its maximal value to its minimal one). The increasing-ρ p_0-majority model is equivalent to the decreasing-ρ $(1-p_0)$-majority model; therefore, for a full study, it is enough to perform *increasing* ρ simulations, and this is what we did. The ρ dependence of ξ for typical values of κ and $p_0 = 2/3$ is depicted in Fig. 2. An interesting fact must be stressed: for $\kappa > 0$ and arbitrary p_0, ξ diverges not only at $\rho = p_0$, but also for ρ approaching its minimal and its maximal values. This is due to the fact that at the $\kappa > 0$ borderline we have $p_{\text{BW}} = p_{\text{WB}} = 0$; hence the clusters maintain the same color. All the divergences of ξ belong to the same universality class. More precisely, $\xi \propto |\rho - p_0|^{-\nu}$ for $\rho - p_0 \to \pm 0$ $(\forall \kappa)$, $\xi \propto (\rho - \rho_{\text{minimal}})^{-\nu}$ for $\rho - \rho_{\text{minimal}} \to +0$, and $\xi \propto (\rho_{\text{maximal}} - \rho)^{-\nu}$ for $\rho - \rho_{\text{maximal}} \to -0$ (for $\kappa > 0$) with $\nu = 1 \pm 0.1$ (see Fig. 3 for a typical example). In Fig. 4 we present some typical examples of the κ dependence of ξ for typical values of p and p_0. The ρ dependence of m for typical values

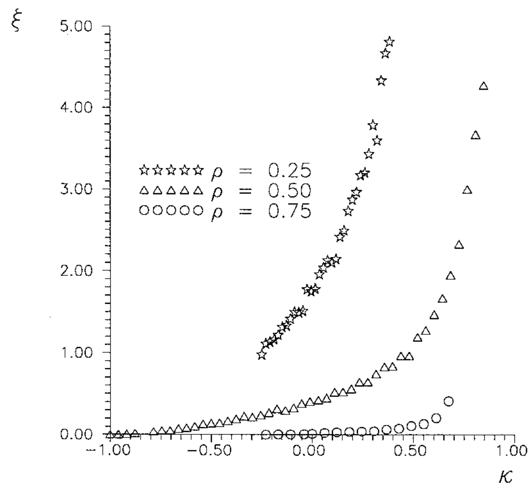

Fig. 4. The κ dependence of the increasing-ρ value of ξ for typical values of ρ and $p_0 = 2/3$. Here $L = 30{,}000$ and $N_0 = 10{,}000$.

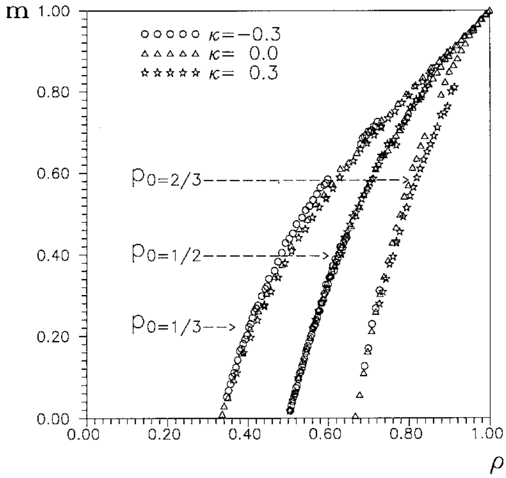

Fig. 5. The ρ dependence (for increasing ρ) of the order parameter m for typical values of p_0 and of the correlation degree κ. Here $L = 30{,}000$ and $N_0 = 10{,}000$.

of p_0 and κ is depicted in Fig. 5. It is quite remarkable that κ has practically no influence on the m versus ρ curves. We obtained in all cases $m \propto (\rho - p_0)^\beta$ $(\rho \to p_0 + 0)$ with $\beta = 1 \pm 0.05$.

3. CONCLUSION

We have discussed the $d = 1$ bichromatic p_0-majority model allowing for isotropic correlations. The present Monte Carlo treatment provides the mean size ξ and the order parameter m as functions of (ρ, κ, p_0). It is shown that all the singularities belong to the same universality class, characterized by $\nu = 1 \pm 0.1$ and $\beta = 1 \pm 0.05$. In other words, analogously with what happens in correlated percolation, the critical behavior is not affected by local correlations. Two more facts deserve special mention: (i) ξ diverges not only at $\rho = p_0$ or $\rho = (1 - p_0)$, but also on the $\kappa > 0$ borderline: (ii) κ has no influence on the (ρ, p_0) dependence of m.

It is worth stressing that critical behavior at finite values of $(1 - \rho)$ is exhibited for a one-dimensional model because the majority check involves counting across arbitrarily large distances, and consequently the effective dimensionality of the problem is in fact infinity. This is of course the reason that mean-field-like exponents are obtained.

The study of the influences of the anisotropy, the dimensionality, and the spatial range of the correlations as well as that of $q \equiv$ (number of colors) being larger than 2 would be welcome.

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